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Phase diagram of the $Z(4)$ ferromagnet in an anisotropic square lattice

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Abstract. Within a real space renormalisation group (RG) scheme, we study the criticality of the ferromagnetic $Z(4)$ model on an anisotropic square lattice. We use an RG cluster which has already proved to be very efficient for the Potts model on the same lattice. The establishment of the RG recurrence relations is greatly simplified through the break-collapse method. The phase diagram (exhibiting ferromagnetic, paramagnetic and nematic-like phases) recovers *all* the available *exact* results, and is believed to be of high precision everywhere. If the model is alternatively thought of as being associated with a particular hierarchical lattice rather than with the square lattice, then it is exact everywhere.

1. Introduction

The $Z(N)$ model unifies in a single framework a large amount of theoretically and experimentally important statistical models (e.g. bond percolation, random resistor networks, spin-$\frac{1}{2}$ Ising, N-state Potts, clock and classical XY models) which are recovered as particular cases. It has attracted, during recent years, a certain amount of effort (Wu and Wang 1976, Elitzur *et al* 1979, Savit 1980, Cardy 1980, Alcaraz and Köberle 1980, 1981, Rujan *et al* 1981, Alcaraz and Tsallis 1982, Baltar *et al* 1984, Mariz *et al* 1985), mainly addressing the square lattice, whose study is simplified because of self-duality. The $Z(N)$ model coincides with the N-state Potts model up to $N=3$, and starts to be more general (more than one coupling constant) at $N=4$, which is the case presently addressed (two coupling constants). The phase diagram of the $Z(4)$ ferromagnet in the square lattice is known to present three phases, namely the paramagnetic (P; $Z(4)$ symmetry), the nematic-like or intermediate (I; $Z(2)$ symmetry) and the ferromagnetic (F; completely broken symmetry) phases. The full phase diagram is constituted by second- or higher-order phase transitions. For the *isotropic* square lattice, the P-F critical line is completely determined by self-duality arguments; furthermore, duality strictly relates the analytically still unknown I-F and I-P lines (although a numerically quite precise determination has been recently undertaken by Mariz *et al* (1985)). The P-F, I-F and I-P lines join at a multicritical point, which is precisely the four-state Potts ferromagnet critical point.

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For the *anisotropic* square lattice (not necessarily equivalent X and Y crystalline axes, each one of which carries two coupling constants) the situation is as follows. The P-F critical frontier (critical volume in a four-dimensional parameter space) is invariant under duality *but* its points are not in general self-dual, and therefore duality arguments are not sufficient for establishing its analytical expression. The I-F and I-P critical volumes still transform, through duality, one into the other. The P-F, I-F and I-P critical volumes join at a multicritical surface, one line of which corresponds to the *anisotropic* square lattice four-state Potts ferromagnetic critical line.

The criticality of the $Z(4)$ ferromagnet on the *isotropic* square lattice has been recently studied (Mariz *et al* 1985) within a real space renormalisation group (RG) formalism based on the well known self-dual Wheatstone bridge cluster; that treatment recovers all the available *exact* results for the corresponding phase diagram, and is on the whole quite satisfactory. Along similar lines, we discuss, in the present paper, the criticality corresponding to the *anisotropic* square lattice; to do so we use a different self-dual cluster, particularly well adapted to this more general situation, and which has already proved its efficiency for the Potts model (de Oliveira and Tsallis 1982).

In § 2, we introduce the model and the RG formalism, in § 3 we present the main results and we finally conclude in § 4.

2. Model and RG formalism

A convenient form for the $Z(4)$ (symmetric Ashkin-Teller model) ferromagnet (dimensionless) Hamiltonian is the following (Alcaraz and Tsallis 1982):

$$\frac{\mathcal{H}}{k_B T} = \sum_{\langle i,j\rangle_x} [K_1^x - K_1^x(\sigma_i\sigma_j + \tau_i\tau_j) - 2K_2^x\sigma_i\sigma_j\tau_i\tau_j] + \sum_{\langle i,j\rangle_y} [K_1^y - K_1^y(\sigma_i\sigma_j + \tau_i\tau_j) - 2K_2^y\sigma_i\sigma_j\tau_i\tau_j] \qquad (1)$$

where T is the temperature, $\langle i,j\rangle_x$ and $\langle i,j\rangle_y$ run over all the pairs of first-neighbouring (respectively along the x and y axes) sites on a square lattice, $\sigma_i = \pm 1$, $\tau_i = \pm 1 (\forall i)$, $K_1^x \geq 0$, $K_1^y \geq 0$, $K_1^x + 2K_2^x \geq 0$ and $K_1^y + 2K_2^y \geq 0$ (the dimensionless coupling constants K are related to the corresponding dimensional ones through $K \equiv J/k_B T$). Let us also introduce the operationally convenient variables (*vector transmissivity*, Alcaraz and Tsallis 1982), $\boldsymbol{t}^x \equiv (1, t_1^x, t_2^x, t_3^x)$ and $\boldsymbol{t}^y \equiv (1, t_1^y, t_2^y, t_3^y)$ through

$$t_1^\gamma = t_3^\gamma \equiv \frac{1 - \exp(-4K_1^\gamma)}{1 + 2\exp[-2(K_1^\gamma + 2K_2^\gamma)] + \exp(-4K_1^\gamma)} \qquad (\gamma = x, y) \qquad (2a)$$

and

$$t_2^\gamma \equiv \frac{1 - 2\exp[-2(K_1^\gamma + 2K_2^\gamma)] + \exp(-4K_1^\gamma)}{1 + 2\exp[-2(K_1^\gamma + 2K_2^\gamma)] + \exp(-4K_1^\gamma)} \qquad (\gamma = x, y). \qquad (2b)$$

This vector transmissivity generalises the scalar one used by Tsallis and Levy (1981) for the Potts model. The Hamiltonian (1) contains several interesting particular cases, namely the four-state Potts model ($K_1^\gamma = 2K_2^\gamma$, hence $t_1^\gamma = t_2^\gamma$) as well as three versions of the spin-$\frac{1}{2}$ Ising model (Ising (1): $K_2^\gamma = 0$, hence $t_2^\gamma = (t_1^\gamma)^2$; Ising (2): $K_1^\gamma = 0$, hence $t_1^\gamma = 0$; Ising (3): $K_2^\gamma = \infty$, hence $t_2^\gamma = 1$).

Let us now establish relationships we shall be needing later on. Consider a series (parallel) array of two bonds with transmissivities $t^{(1)}$ and $t^{(2)}$; the equivalent transmissivity $t^{(s)}(t^{(p)})$ is given by (Alcaraz and Tsallis 1982, Mariz *et al* 1985)

$$t_r^{(s)} = t_r^{(1)} t_r^{(2)} \qquad (r=1,2) \qquad \text{(series)} \tag{3}$$

and

$$t_1^{(p)} = \frac{t_1^{(1)} + t_1^{(2)} + t_1^{(1)} t_2^{(2)} + t_1^{(2)} t_2^{(1)}}{1 + 2t_1^{(1)} t_1^{(2)} + t_2^{(1)} t_2^{(2)}} \qquad \text{(parallel)} \tag{4a}$$

$$t_2^{(p)} = \frac{t_2^{(1)} + t_2^{(2)} + 2t_1^{(1)} t_1^{(2)}}{1 + 2t_1^{(1)} t_1^{(2)} + t_2^{(1)} t_2^{(2)}} \qquad \text{(parallel)}. \tag{4b}$$

Equations (4) can be conveniently rewritten as follows:

$$t_r^{(p)D} = t_r^{(1)D} t_r^{(2)D} \qquad (r=1,2) \tag{5}$$

where the *dual* transmissivity t^D is defined by

$$t_1^D \equiv \frac{1-t_2}{1+2t_1+t_2} \tag{6a}$$

$$t_2^D \equiv \frac{1-2t_1+t_2}{1+2t_1+t_2}. \tag{6b}$$

We can now go back to the anisotropic square lattice. To construct the RG recurrence relations (in the $(t_1^x, t_2^x, t_1^y, t_2^y)$ space, for instance), we follow along the lines of the Potts model treatment of de Oliveira and Tsallis (1982), and renormalise the cluster (two-rooted graph) indicated in figure 1(*b*) into the single bond indicated in figure 1(*a*). To be more explicit, we construct the present RG in such a way as to preserve the two-body correlation functions (such a procedure is very efficient even for quantum systems; see for instance Caride *et al* (1983)), i.e. (along the x axis)

$$\exp(-\mathscr{H}'_{12}) = \operatorname*{Tr}_{3,4,5,6} \exp(-\mathscr{H}_{123456}) \tag{7}$$

where the renormalised (dimensionless) Hamiltonian $\mathscr{H}'_{12}$ is given (except for an

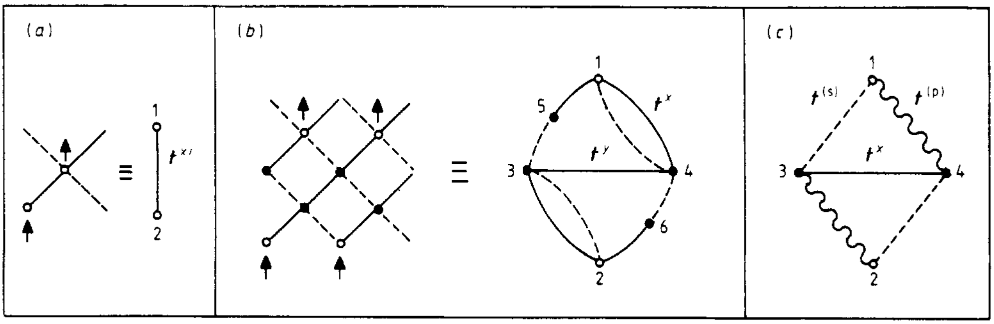

Figure 1. Two-rooted graphs associated with the x axis RG recursive relations (the y axis ones are completely analogous), obtained by renormalising cluster (*b*) into cluster (*a*) (the full line and broken line represent respectively the x and y bonds of the square lattice; the arrows indicate the 'entrances' and 'exits' of the clusters; ● and ○ respectively denote internal and terminal sites). Graph (*c*) is equivalent to graph (*b*) with dotted line and wavy line representing respectively series and parallel arrays of the x and y bonds.

additive constant) by

$$\mathscr{H}'_{12} = K_1^{x'} - K_1^{x'}(\sigma_1\sigma_2 + \tau_1\tau_2) - 2K_2^{x'}\sigma_1\sigma_2\tau_1\tau_2 \quad (8)$$

and the cluster (dimensionless) Hamiltonian $\mathscr{H}_{123456}$ is given by

$$\begin{aligned}\mathscr{H}_{123456} = 5K_1^x &- K_1^x(\sigma_1\sigma_5 + \tau_1\tau_5 + \sigma_1\sigma_4 + \tau_1\tau_4 + \sigma_3\sigma_4 + \tau_3\tau_4 + \sigma_2\sigma_3 + \tau_2\tau_3 + \sigma_2\sigma_6 + \tau_2\tau_6)\\ &-2K_2^x(\sigma_1\sigma_5\tau_1\tau_5 + \sigma_1\sigma_4\tau_1\tau_4 + \sigma_3\sigma_4\tau_3\tau_4 + \sigma_2\sigma_3\tau_2\tau_3 + \sigma_2\sigma_6\tau_2\tau_6)\\ &+4K_1^y - K_1^y(\sigma_1\sigma_4 + \tau_1\tau_4 + \sigma_3\sigma_5 + \tau_3\tau_5 + \sigma_4\sigma_6 + \tau_4\tau_6 + \sigma_2\sigma_3 + \tau_2\tau_3)\\ &-2K_2^y(\sigma_1\sigma_4\tau_1\tau_4 + \sigma_3\sigma_5\tau_3\tau_5 + \sigma_4\sigma_6\tau_4\tau_6 + \sigma_2\sigma_3\tau_2\tau_3). \end{aligned} \quad (9)$$

We immediately see that the graph indicated in figure 1(b) is equivalent to that indicated in figure 1(c) where $\boldsymbol{t}^{(\mathrm{s})}$ and $\boldsymbol{t}^{(\mathrm{p})}$ are respectively given by equation (3) and equations (4) with $\boldsymbol{t}^{(1)} = \boldsymbol{t}^x$ and $\boldsymbol{t}^{(2)} = \boldsymbol{t}^y$. The next step is to calculate the transmissivity (identified with $\boldsymbol{t}^{x'}$) of the graph indicated in figure 1(c). We perform this through *the break-collapse method* (BCM), introduced by Tsallis and Levy (1981) for the Potts model and recently extended by Mariz *et al* (1985) to the $Z(4)$ model (see Tsallis (1985) for a review). The transmissivity $t_1^{x'}$ is given by

$$t_1^{x'} = \frac{N_1(t_1^x, t_2^x; t_1^y, t_2^y)}{D(t_1^x, t_2^x; t_1^y, t_2^y)} \quad (10a)$$

and

$$t_2^{x'} = \frac{N_2(t_1^x, t_2^x; t_1^y, t_2^y)}{D(t_1^x, t_2^x; t_1^y, t_2^y)} \quad (10b)$$

where N_1, N_2 and D are to be determined. To do this we shall operate on the central bond of figure 1(c) (in fact, we could just as well choose any other bond), and obtain the *broken* ($t_1^x = t_2^x = 0$), the *collapsed* ($t_1^x = t_2^x = 1$) and the *pre-collapsed* ($t_1^x = 0$, $t_2^x = 1$) graphs, respectively indicated in figures 2(a)-(c). Let us note $\boldsymbol{t}^{\mathrm{bb}} \equiv (t_1^{\mathrm{bb}}, t_2^{\mathrm{bb}}) \equiv (N_1^{\mathrm{bb}}/D^{\mathrm{bb}}, N_2^{\mathrm{bb}}/D^{\mathrm{bb}})$, $\boldsymbol{t}^{\mathrm{cc}} \equiv (t_1^{\mathrm{cc}}, t_2^{\mathrm{cc}}) \equiv (N_1^{\mathrm{cc}}/D^{\mathrm{cc}}, N_2^{\mathrm{cc}}/D^{\mathrm{cc}})$ and $\boldsymbol{t}^{\mathrm{bc}} \equiv (t_1^{\mathrm{bc}}, t_2^{\mathrm{bc}}) \equiv (N_1^{\mathrm{bc}}/D^{\mathrm{bc}}, N_2^{\mathrm{bc}}/D^{\mathrm{bc}})$, the transmissivities respectively associated with the graphs of

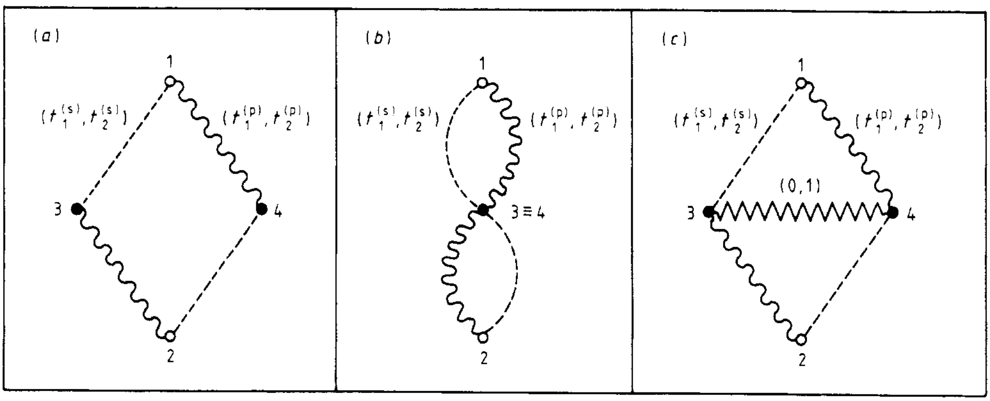

Figure 2. Two-rooted graphs obtained by breaking ($t_1^x = t_2^x = 0$; graph (a)), collapsing ($t_1^x = t_2^x = 1$; graph (b)) and pre-collapsing ($t_1^x = 1 - t_2^x = 0$; graph (c)) the $\boldsymbol{t}^x$ bond of the graph of figure 1(c). Sawtooth line represents a pre-collapsed bond.

figure 2. The quantities N_1, N_2 and D we are looking for are given (BCM; Mariz *et al* 1985) by

$$N_r = (1-t_2^x)N_r^{\mathrm{bb}} + t_1^x N_r^{\mathrm{cc}} + (t_2^x - t_1^x)N_r^{\mathrm{bc}} \qquad (r=1,2) \tag{11}$$

and

$$D = (1-t_2^x)D^{\mathrm{bb}} + t_1^x D^{\mathrm{cc}} + (t_2^x - t_1^x)D^{\mathrm{bc}} \tag{12}$$

and consequently the knowledge of N_r^{bb}, D^{bb}, N_r^{cc}, D^{cc}, N_r^{bc} and D^{bc} enables the calculation of N_r and D.

The transmissivities $\boldsymbol{t}^{\mathrm{bb}}$ and $\boldsymbol{t}^{\mathrm{cc}}$ are easily calculated (by using the series and parallel algorithms expressed in equations (3) and (4)) as the respective graphs (figures 2(*a*) and 2(*b*)) are reducible in series and parallel operations. The transmissivity $\boldsymbol{t}^{\mathrm{bc}}$ is more complex, and has to be further reduced through the BCM (recursive use of the algorithm expressed in equations (11) and (12)). All graphs reducible in series and parallel operations are straightforwardly calculated. Only one graph resists until the very last step, and this graph exclusively contains (0, 1) bonds: the transmissivity of such a graph itself satisfies $t_1 = 0$ and $t_2 = 1$. The problem is thus completely solved. We obtain

$$\begin{aligned} N_1(t_1^x, t_2^x; t_1^y, t_2^y) &\\ = 2t_1^{(s)}t_1^{(p)} &+ 2t_1^{(s)}t_1^{(p)}t_2^{(s)}t_2^{(p)} + [(t_1^{(s)})^2 + (t_1^{(p)})^2]t_1^x + 2(t_1^{(s)})^2 t_1^x t_2^{(p)} \\ &+ 2(t_1^{(p)})^2 t_1^x t_2^{(s)} + [(t_1^{(s)})^2(t_2^{(p)})^2 + (t_1^{(p)})^2(t_2^{(s)})^2]t_1^x + 2(t_2^{(s)} + t_2^{(p)})t_1^{(s)}t_1^{(p)}t_2^x \end{aligned} \tag{13}$$

$$\begin{aligned} N_2(t_1^x, t_2^x; t_1^y, t_2^y) &\\ = 2t_2^{(s)}t_2^{(p)} &+ 2(t_1^{(s)})^2(t_1^{(p)})^2 + [(t_2^{(s)})^2 + (t_2^{(p)})^2]t_2^x \\ &+ 2(t_1^{(s)})^2(t_1^{(p)})^2 t_2^x + 4(t_2^{(s)} + t_2^{(p)})t_1^{(s)}t_1^{(p)}t_1^x \end{aligned} \tag{14}$$

and

$$\begin{aligned} D(t_1^x, t_2^x; t_1^y, t_2^y) = 1 &+ (t_2^{(s)})^2(t_2^{(p)})^2 + 2(t_1^{(s)})^2(t_1^{(p)})^2 + 4t_1^{(s)}t_1^{(p)}t_1^x \\ &+ 4t_1^{(s)}t_1^{(p)}t_1^x t_2^{(s)}t_2^{(p)} + 2t_2^{(s)}t_2^{(p)}t_2^x + 2(t_1^{(s)})^2(t_1^{(p)})^2 t_2^x. \end{aligned} \tag{15}$$

Summarising, the RG recursive relations are as follows:

$$t_1^{x'} = \frac{N_1(t_1^x, t_2^x; t_1^y, t_2^y)}{D(t_1^x, t_2^x; t_1^y, t_2^y)} \equiv f_1(t_1^x, t_2^x; t_1^y, t_2^y) \tag{16}$$

$$t_2^{x'} = \frac{N_2(t_1^x, t_2^x; t_1^y, t_2^y)}{D(t_1^x, t_2^x; t_1^y, t_2^y)} \equiv f_2(t_1^x, t_2^x; t_1^y, t_2^y) \tag{17}$$

$$t_1^{y'} = f_1(t_1^y, t_2^y; t_1^x, t_2^x) \tag{18}$$

and

$$t_2^{y'} = f_2(t_1^y, t_2^y; t_1^x, t_2^x) \tag{19}$$

where in the last two equations we have taken into account the $x \rightleftarrows y$ invariance of the square lattice. This set of four equations completely determines the flow in the $(t_1^x, t_2^x, t_1^y, t_2^y)$ space, and through it the phase diagram as well as the universality classes of our system.

In order to express the results in more familiar variables, it is convenient to introduce the following definitions:

$$\tau \equiv k_B T / J_1^x = 1/K_1^x \tag{20}$$

$$\alpha_1 \equiv J_1^y / J_1^x = K_1^y / K_1^x \tag{21}$$

$$\alpha_2 \equiv \frac{J_1^y + 2J_2^y}{J_1^x + 2J_2^x} = \frac{K_1^y + 2K_2^y}{K_1^x + 2K_2^x} \tag{22}$$

$$\beta^x \equiv \frac{J_1^x + 2J_2^x}{J_1^x} = \frac{K_1^x + 2K_2^x}{K_1^x} \tag{23}$$

$$\beta^y \equiv \frac{J_1^y + 2J_2^y}{J_1^y} = \frac{K_1^y + 2K_2^y}{K_1^y} \tag{24}$$

where we remark that the following relationship holds: $\alpha_2/\alpha_1 = \beta^y/\beta^x$. Our phase diagram can also be conveniently expressed in the $(\tau, \alpha_1, \beta^x, \beta^y)$ space. Finally let us also introduce a convenient variable (Alcaraz and Tsallis 1982) through the following definition:

$$s(t_1, t_2) \equiv \ln(1 + 2t_1 + t_2)/\ln 4. \tag{25}$$

Notice an interesting property, namely

$$s^D(t_1, t_2) \equiv s(t_1^D, t_2^D) = 1 - s(t_1, t_2). \tag{26}$$

3. Results

The RG flow exhibits three trivial (fully stable) fixed points, namely $(t_1^x, t_2^x, t_1^y, t_2^y) = (0, 0, 0, 0)$ (characterising the P phase), $(1, 1, 1, 1)$ (characterising the F phase) and $(0, 1, 0, 1)$ (characterising the I phase). The P–F critical three-dimensional volume (in the four-dimensional parameter space) is preserved through duality (and consequently through our RG which is constructed on a self-dual cluster), i.e. if $(t_1^x, t_2^x, t_1^y, t_2^y)$ belongs to this volume, then $(t_1^{xD}, t_2^{xD}, t_1^{yD}, t_2^{yD})$ given by

$$t_1^{xD} \equiv \frac{1 - t_2^x}{1 + 2t_1^x + t_2^x} \tag{27}$$

$$t_2^{xD} \equiv \frac{1 - 2t_1^x + t_2^x}{1 + 2t_1^x + t_2^x} \tag{28}$$

$$t_1^{yD} \equiv \frac{1 - t_2^y}{1 + 2t_1^y + t_2^y} \tag{29}$$

$$t_2^{yD} \equiv \frac{1 - 2t_1^y + t_2^y}{1 + 2t_1^y + t_2^y} \tag{30}$$

also belongs to it. However, excepting special cases, the points of this critical volume are not self-dual, i.e. in general $(t_1^x, t_2^x, t_1^y, t_2^y) \neq (t_1^{xD}, t_2^{xD}, t_1^{yD}, t_2^{yD})$. Due to this fact, duality arguments are not sufficient for establishing the analytical expression of the P–F critical volume. Two regions of this volume are constituted by self-dual points. These regions are the following.

(i) The '*anisotropic*' *self-dual surface*, determined by

$$t_1^x = t_1^{yD} \tag{31a}$$

$$t_2^x = t_2^{yD} \tag{31b}$$

which imply $s^x + s^y = 1$ with $s^x \equiv s(t_1^x, t_2^x)$ and $s^y \equiv s(t_1^y, t_2^y)$ where we have used definition (25). This surface contains the anisotropic Potts ferromagnet critical line for $t_1^x = t_2^x$ and $t_1^y = t_2^y$, as well as the anisotropic Ising (1) ferromagnet critical line for $t_2^x = (t_1^x)^2$ and $t_2^y = (t_1^y)^2$ (both critical lines are exactly recovered within the present RG).

(ii) The '*isotropic*' *self-dual surface*, determined by

$$t_2^x = 1 - 2t_1^x \tag{32a}$$

$$t_2^y = 1 - 2t_1^y \tag{32b}$$

or equivalently by

$$s^x = s^y = \tfrac{1}{2}. \tag{33}$$

On the intersection between the isotropic and anisotropic self-dual surfaces lies the already known P-F critical line of the $Z(4)$ ferromagnet in an isotropic square lattice ($t_1^x = t_1^y$ and $t_2^x = t_2^y$) (see for instance Mariz *et al* 1985). The whole situation is depicted in figure 3. A point which belongs to the P-F critical volume but does not lie on any of the self-dual surfaces is transformed, through duality (equations (27)-(30)), on another point which *also* belongs to the P-F critical volume and which is located on the 'other side' with respect to the anisotropic self-dual surface as well as with respect to the isotropic self-dual surface (such an operation transforms (s^x, s^y) into $(1 - s^x, 1 - s^y)$).

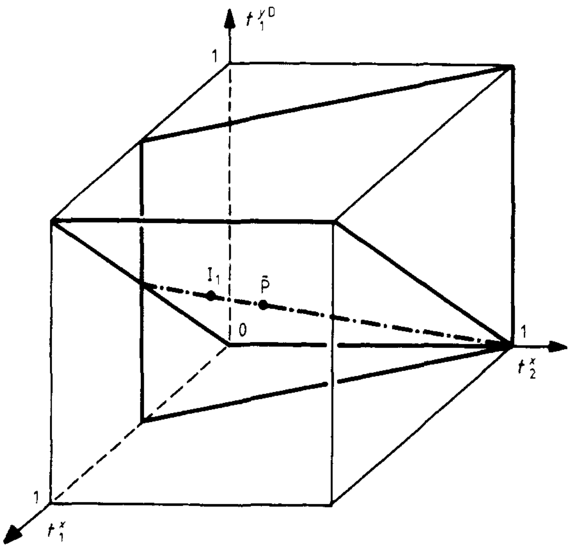

Figure 3. Cuts of the 'isotropic' and 'anisotropic' self-dual surfaces with the $t_2^{yD} = 0$ volume in the four-dimensional $(t_1^x, t_2^x, t_1^{yD}, t_2^{yD})$ space (or, equivalently, the $(t_1^x, t_2^x, t_1^y, t_2^y)$ space). The 'isotropic' ('anisotropic') surface is determined by $t_2^x + 2t_1^x = t_2^y + 2t_1^y = 1$ ($t_1^x = t_1^{yD}$ and $t_2^x = t_2^{yD}$), and is so-called because it satisfies $s^x = s^y = \frac{1}{2}(s^x + s^y = 1)$. $\tilde{P}$ and I_1 respectively indicate the Potts and Ising (1) critical points, which lie on the isotropic $Z(4)$ self-dual line (to which belongs its P-F critical line).

The P-F critical volume bifurcates at a multicritical surface, into two critical volumes, namely the I-P and I-F ones. These two volumes are transformed into each other through duality (equations (27)-(30)) and, except for the Ising limits and special parts of the bifurcation multicritical surface, do not contain regions constituted by self-dual points. Their analytical description is therefore far from trivial.

The analytical expression of the multicritical surface itself is unknown; nevertheless it is easy to verify that the anisotropic Potts ferromagnet critical line ($t_1^x = t_2^x = t_1^{yD} = t_2^{yD}$) belongs to it.

The RG flow within the critical volumes is as follows.

(i) Almost all points of the P-F critical volume are attracted by the $d=2$ isotropic Ising (1) fixed point ($t_1^x = t_1^y = \sqrt{t_2^x} = \sqrt{t_2^y} = \sqrt{2}-1$), and therefore belong to the corresponding universality class (the present treatment yields for the correlation length

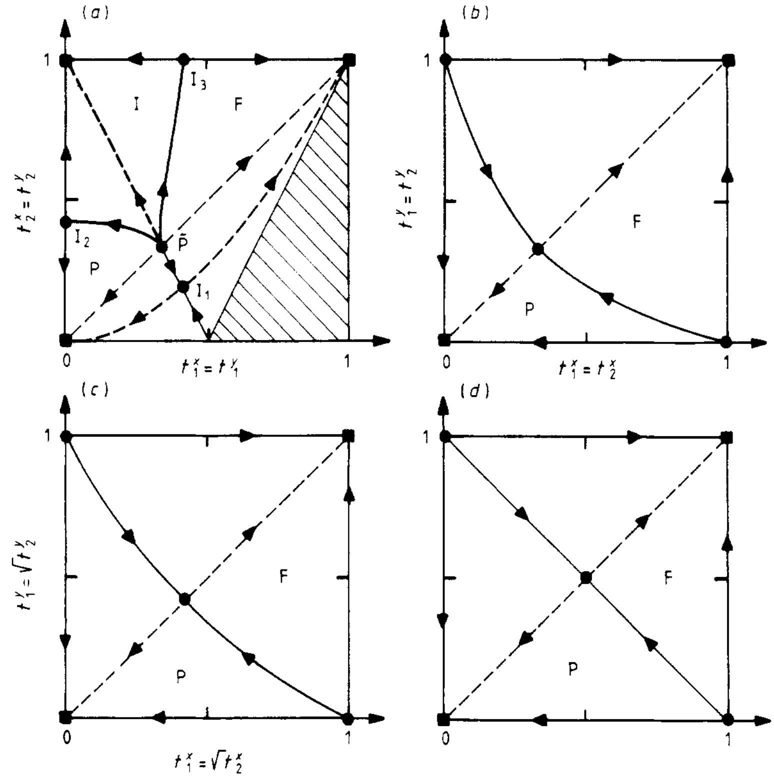

Figure 4. RG flow in the main invariant subspaces: (a) isotropic $Z(4)$ model ($\tilde{\mathrm{P}}$, I_1, I_2 and I_3 respectively denote the Potts, Ising (1), Ising (2) and Ising (3) critical points; the broken area is unphysical); (b) anisotropic four-state Potts model; (c) anisotropic Ising (1) model. P, F and I respectively indicate the paramagnetic, ferromagnetic and intermediate (nematic-like) phases; ● and ■ respectively represent unstable and fully stable fixed points; (d) can indistinctively represent the flow in (b) (by respectively choosing $s(t_1^x, t_1^x)$ and $s(t_1^y, t_1^y)$ as abcissa and ordinate), as well as that in (c) (by respectively choosing $s(t_1^x, (t_1^x)^2)$ and $s(t_1^y, (t_1^y)^2)$ as abcissa and ordinate); it can also represent the flow associated with the Ising (2) model (by respectively choosing $2s(0, t_2^x)$ and $2s(0, t_2^y)$ as abcissa and ordinate; $t_1^x = t_1^y = 0$), as well as that of the Ising (3) model (by respectively choosing $2s(0, t_1^x)$ and $2s(0, t_1^y)$ as abcissa and ordinate; $t_2^x = t_2^y = 1$). (b), (c) and (d): the (1, 0) and (0, 1) fixed points are the $d=1$ points.

critical exponent the value $\nu_{\text{Ising}} = \ln 3/\ln(29/13) \simeq 1.369$, to be compared with the exact value $\nu_{\text{Ising}}^{\text{exact}} = 1$; we recall that the present RG linear scale factor b equals 3 (shortest distance between roots of the graph; see Melrose (1983a, b)); this result is exact for the hierarchical lattice defined by the recursive graph transformation indicated in figures 1(a) and (b) (and the corresponding one for the y axis), but is incorrect for the Bravais square lattice, which is known (Kohmoto *et al* 1981) to be associated, for

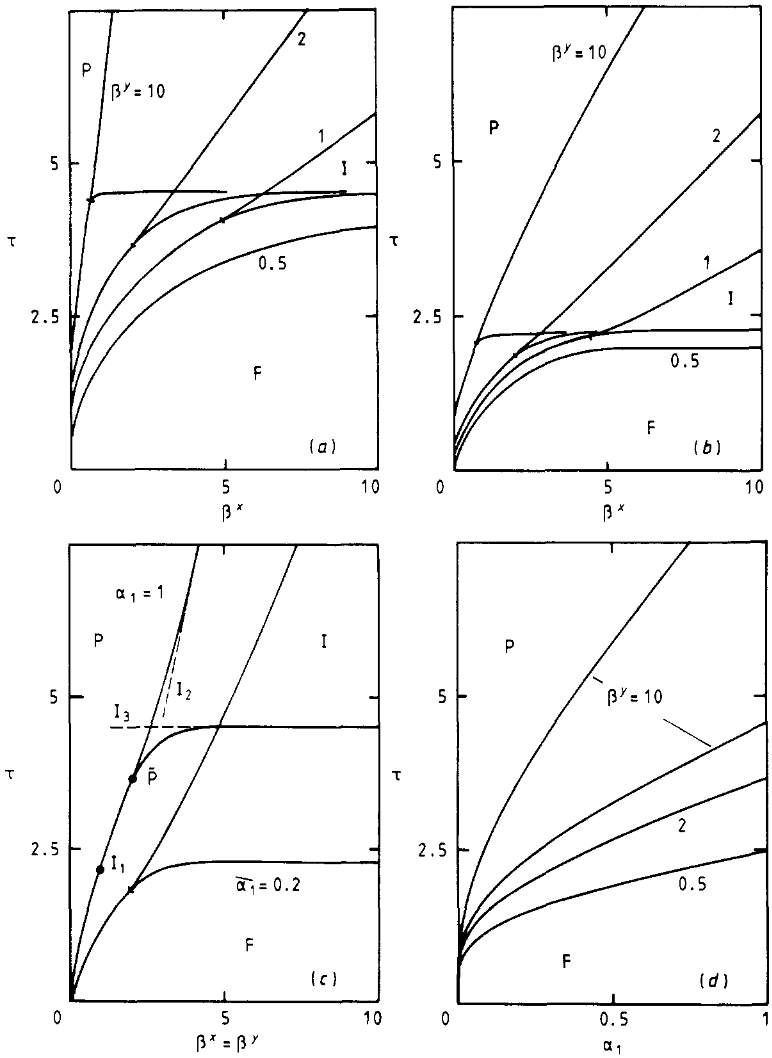

Figure 5. Typical cuts of the anisotropic $Z(4)$ ferromagnet phase diagram. $\tau \equiv k_B T/J_1^x$ is the reduced temperature; α_1, β^x and β^y are defined in the text. The ferromagnetic (F), intermediate (I) and paramagnetic (P) phases respectively appear at low, intermediate and high temperatures. The I phase always disappears for β^x and β^y low enough. $\beta^x = \beta^y = 1$ and 2 respectively recover the anisotropic Ising (1) (I_1) and four-state Potts ($\tilde{P}$) critical lines. In the limit of high β^x and/or β^y, the I-P and I-F phase boundary asymptotically and respectively yield the anisotropic Ising (2) (I_2) and Ising (3) (I_3) critical lines. The $\alpha_1 = \beta^y/\beta^x = 1$ case corresponds to the isotropic $Z(4)$ ferromagnet. (a) $\alpha_1 = 1$, (b) $\alpha_1 = 0.2$, (c) $\beta^x = \beta^y$, (d) $\beta^x = 2$.

the isotropic case, with a continuously varying set of universality classes (this discrepancy could possibly disappear in the limit of increasingly large RG clusters).

(ii) Almost all points of the I–P and I–F critical volumes are respectively attracted by the $d=2$ isotropic Ising (2) and Ising (3) fixed points (respectively at $t_1^x=t_1^y=0$ and $t_2^x=t_2^y=\sqrt{2}-1$, and at $t_2^x=t_2^y=1$ and $t_1^x=t_1^y=\sqrt{2}-1$), and therefore belong to the $d=2$ Ising universality class, as expected from symmetry arguments.

(iii) Almost all points of the bifurcation multicritical surface flow towards the $d=2$ isotropic four-state Potts fixed point ($t_1^x=t_2^x=t_1^y=t_2^y=\frac{1}{3}$), and therefore belong to the corresponding universality class (we obtain $\nu_{\text{Potts}}=\ln 3/\ln(\frac{2193}{857})\simeq 1.169$, to be compared with the exact value (den Nijs 1979), $\nu_{\text{Potts}}^{\text{exact}}=\frac{2}{3}$ for the Bravais square lattice).

(iv) All points of the P–F, I–P and I–F critical volumes not yet covered by points (i)–(iii) either correspond to one or the other $d=1$ fixed points $[(t_1^x, t_2^x, t_1^y, t_2^y)=(1,1,0,0), (0,0,1,1), (0,1,0,0), (0,0,0,1), (1,1,0,1), (0,1,1,1)]$ and therefore belong to the standard N-state Potts one-dimensional universality class (we obtain $\nu_{1\text{D}}=\nu_{1\text{D}}^{\text{exact}}=1$), or correspond to new unstable fixed points at the boundary of the physical region (*real* coupling constants), e.g. $(t_1^x, t_2^x, t_1^y, t_2^y)=(\frac{1}{2},0,0,1)$, $(0,1,\frac{1}{2},0)$.

The previous statements concerning the RG flow are illustrated in figure 4 for a few interesting invariant subspaces. Typical cuts of the full phase diagram are represented in figure 5 in the $(\tau, \alpha_1, \beta^x, \beta^y)$ variables.

4. Conclusion

The criticality of the $Z(4)$ ferromagnet in an anisotropic square lattice has been studied within a real space renormalisation group (RG) which preserves two-body correlation functions. To construct the RG recursive relations we have adopted a cluster (two-rooted graph) which has already proved its efficiency for the N-state Potts ferromagnet in the same Bravais lattice, and which presents several interesting features: (i) it is self-dual and reproduces consequently all the available *exact* results concerning the still unknown critical frontier associated with the square lattice, itself self-dual; (ii) it presents a peculiar x- and y-bond topological structure which, in the high anisotropy limit, exactly recovers the linear chain, therefore exhibiting $d=1\leftrightarrow d=2$ crossovers which are consistent with the symmetry-based expectations; (iii) it generates a hierarchical lattice whose fractal dimensionality $d_f=2$ ($d_f\equiv\ln(\text{aggregation number})/\ln b=\ln 9/\ln 3=2$), coincident with that of the Bravais lattice which it is intended to approach.

In spite of the relative complexity of the cluster (six spins and nine bonds) and of the model (four states per spin and four coupling constants) which yield $4^6=4096$ different configurations, it has been possible, through the use of the break-collapse method which greatly simplifies the analytical operational task, to establish by hand the RG explicit recursive relations with little effort. This set of equations enables the quick numerical calculation of an arbitrary point of the phase diagram (three critical volumes separating the paramagnetic, intermediate nematic-like and ferromagnetic phases in a four-dimensional parameter space) as well as the qualitative discussion of the main special features (role played by the duality transformation, etc). All these results are either known to be exact (e.g. the critical lines of the anisotropic Ising and Potts ferromagnet as well as the para-ferro critical line of the isotropic $Z(4)$ model), or believed (by us) to be so (e.g. equations (32)), or high precision ones everywhere for the anisotropic square lattice. The model being classical (in the sense that all relevant observables commutate) and no proliferation of the coupling constants taking place, the whole phase diagram (as well as the critical exponents, which exhibit non-neglectable discrepancies with those corresponding to the Bravais square lattice) is exact for the hierarchical lattice generated by the recursive graph transformation.

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References

Alcaraz F C and Köberle R 1980 *J. Phys. A: Math. Gen.* **13** L153
—— 1981 *J. Phys. A: Math. Gen.* **14** 1169
Alcaraz F C and Tsallis C 1982 *J. Phys. A: Math. Gen.* **15** 587
Baltar V L, Carneiro G M, Pol M E and Zagury N 1984 *J. Phys. A: Math. Gen.* **17** 2119
Cardy J L 1980 *J. Phys. A: Math. Gen.* **13** 1507
Caride A O, Tsallis C and Zanette S J 1983 *Phys. Rev. Lett.* **51** 145 (erratum **51** 616)
de Oliveira P M C and Tsallis C 1982 *J. Phys. A: Math. Gen.* **15** 2865
den Nijs M P M 1979 *Physica* **95A** 449
Elitzur S, Pearson R B and Shigemitsu J 1979 *Phys. Rev.* D **19** 3698
Kohmoto M, den Nijs M P M and Kadanoff L P 1981 *Phys. Rev.* B **24** 5229
Mariz A, Tsallis C and Fulco P 1985 *Phys. Rev.* B **32** 6055
Melrose J R 1983a *J. Phys. A: Math. Gen.* **16** L407
—— 1983b *J. Phys. A: Math. Gen.* **16** 3077
Rujan P, Williams G O, Frisch H L and Forgâcs G 1981 *Phys. Rev.* B **23** 1362
Savit R 1980 *Rev. Mod. Phys.* **52** 453
Tsallis C 1986 *Phys. Rep.* to appear
Tsallis C and Levy S V F 1981 *Phys. Rev. Lett.* **47** 950
Wu F Y and Wang Y K 1976 *J. Math. Phys.* **17** 439